# Calculus 1
# Workbook Solutions

## Linear approximation and linearization

krista king
MATH

## LINEAR APPROXIMATION

■ 1. Find the linearization of $f(x) = x^3 - 4x^2 + 2x - 3$ at $x = 3$ and use the linearization to approximate $f(3.02)$.

*Solution:*

The linearization formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 3$, so the linearization is $L(x) = f(3) + f'(3)(x - 3)$. Find the pieces that we need for the linearization formula.

$$f(3) = 3^3 - 4(3)^2 + 2(3) - 3 = -6$$

$$f'(x) = 3x^2 - 8x + 2$$

$$f'(3) = 3(3)^2 - 8(3) + 2 = 5$$

Plugging these pieces into the linearization gives

$$L(x) = -6 + 5(x - 3)$$

$$L(x) = -6 + 5x - 15$$

$$L(x) = 5x - 21$$

Use this equation to approximate $f(3.02)$.

$$f(3.02) = 5(3.02) - 21 = -5.9$$

© Krista King Math

■ 2. Find the linearization of $g(x) = \sqrt{8x - 15}$ at $x = 8$ and use the linearization to approximate $f(8.05)$.

*Solution:*

The linearization formula at $x = a$ is $L(x) = g(a) + g'(a)(x - a)$. In this problem, $a = 8$, so the linearization is $L(x) = g(8) + g'(8)(x - 8)$. Find the pieces that we need for the linearization formula.

$$g(8) = \sqrt{8(8) - 15} = \sqrt{49} = 7$$

$$g'(x) = \frac{8}{2\sqrt{8x - 15}} = \frac{4}{\sqrt{8x - 15}}$$

$$g'(8) = \frac{4}{\sqrt{8(8) - 15}} = \frac{4}{\sqrt{49}} = \frac{4}{7}$$

Plugging these pieces into the linearization gives

$$L(x) = 7 + \frac{4}{7}(x - 8)$$

$$L(x) = 7 + \frac{4}{7}x - \frac{32}{7}$$

$$L(x) = \frac{4}{7}x + \frac{17}{7}$$

Use this equation to approximate $g(8.05)$.

$$g(8.05) = \frac{4}{7}(8.05) + \frac{17}{7} = \frac{246}{35} \approx 7.029$$

© Krista King Math

■ 3. Find the linearization of $h(x) = 2e^{x-4} + 6$ at $x = 5$ and use the linearization to approximate $h(5.1)$.

*Solution:*

The linearization formula at $x = a$ is $L(x) = h(a) + h'(a)(x - a)$. In this problem, $a = 5$, so the linearization is $L(x) = h(5) + h'(5)(x - 5)$. Find the pieces that we need for the linearization formula.

$$h(5) = 2e^{5-4} + 6 = 2e + 6$$

$$h'(x) = 2e^{x-4}$$

$$h'(5) = 2e^{5-4} = 2e$$

Plugging these pieces into the linearization gives

$$L(x) = 2e + 6 + 2e(x - 5)$$

$$L(x) = 2e + 6 + 2ex - 10e$$

$$L(x) = 2ex - 8e + 6$$

Use this equation to approximate $h(5.1)$.

$$g(5.1) = 2e(5.1) - 8e + 6 = 10.2e - 8e + 6 = 2.2e + 6 \approx 11.98$$

© Krista King Math

## LINEAR APPROXIMATION TO ESTIMATE A ROOT

■ 1. Use linear approximation to estimate $\sqrt[5]{34}$.

*Solution:*

Let $f(x) = \sqrt[5]{x}$ and $a = 32$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(32) + f'(32)(x - 32)$$

Find the pieces needed for the formula.

$$f(32) = \sqrt[5]{32} = 2$$

$$f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$$

$$f'(x) = \frac{1}{5}(x)^{-\frac{4}{5}} = \frac{1}{5 \cdot \sqrt[5]{x^4}}$$

$$f'(32) = \frac{1}{5 \cdot \sqrt[5]{32^4}} = \frac{1}{80}$$

Then the linear approximation is

$$L(x) = 2 + \frac{1}{80}(x - 32)$$

© Krista King Math

$$L(x) = \frac{1}{80}x + \frac{8}{5}$$

Use this approximation to estimate $\sqrt[5]{34}$.

$$f(34) = \frac{1}{80}(34) + \frac{8}{5} = \frac{81}{40}$$

■ 2. Use linear approximation to estimate $\sqrt[8]{260}$.

*Solution:*

Let $f(x) = \sqrt[8]{x}$ and $a = 256$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(256) + f'(256)(x - 256)$$

Find the pieces needed for the formula.

$$f(256) = \sqrt[8]{256} = 2$$

$$f(x) = \sqrt[8]{x} = x^{\frac{1}{8}}$$

$$f'(x) = \frac{1}{8}(x)^{-\frac{7}{8}} = \frac{1}{8\sqrt[8]{x^7}}$$

$$f'(256) = \frac{1}{8\sqrt[8]{256^7}} = \frac{1}{1{,}024}$$

© Krista King Math

Then the linear approximation is

$$L(x) = 2 + \frac{1}{1{,}024}(x - 256)$$

$$L(x) = \frac{1}{1{,}024}x + \frac{7}{4}$$

Use this approximation to estimate $\sqrt[8]{260}$.

$$f(260) = \frac{1}{1{,}024}(260) + \frac{7}{4} = \frac{513}{256} \approx 2.0039$$

■ 3. Use linear approximation to estimate $\sqrt[4]{85}$.

*Solution:*

Let $f(x) = \sqrt[4]{x}$ and $a = 81$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(81) + f'(81)(x - 81)$$

Find the pieces needed for the formula.

$$f(81) = \sqrt[4]{81} = 3$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

© Krista King Math

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(81) = \frac{1}{4\sqrt[4]{81^3}} = \frac{1}{108}$$

Then the linear approximation is

$$L(x) = 3 + \frac{1}{108}(x - 81)$$

$$L(x) = \frac{1}{108}x + \frac{9}{4}$$

Use this approximation to estimate $\sqrt[4]{85}$.

$$f(85) = \frac{1}{108}(85) + \frac{9}{4} = \frac{82}{27} \approx 3.037$$

■ 4. Use linear approximation to estimate $\sqrt[4]{615}$.

*Solution:*

Let $f(x) = \sqrt[4]{x}$ and $a = 625$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(625) + f'(625)(x - 625)$$

Find the pieces needed for the formula.

© Krista King Math

$$f(625) = \sqrt[4]{625} = 5$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(625) = \frac{1}{4\sqrt[4]{625^3}} = \frac{1}{500}$$

Then the linear approximation is

$$L(x) = 5 + \frac{1}{500}(x - 625)$$

$$L(x) = \frac{1}{500}x + \frac{15}{4}$$

Use this approximation to estimate $\sqrt[4]{615}$.

$$f(615) = \frac{1}{500}(615) + \frac{15}{4} = \frac{249}{50} \approx 4.98$$

■ 5. Use linear approximation to estimate $\sqrt{95}$.

*Solution:*

Let $f(x) = \sqrt{x}$ and $a = 100$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

© Krista King Math

$$L(x) = f(100) + f'(100)(x - 100)$$

Find the pieces needed for the formula.

$$f(100) = \sqrt{100} = 10$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

Then the linear approximation is

$$L(x) = 10 + \frac{1}{20}(x - 100)$$

$$L(x) = \frac{1}{20}x + 5$$

Use this approximation to estimate $\sqrt{95}$.

$$f(95) = \frac{1}{20}(95) + 5 = \frac{39}{4} \approx 9.75$$

■ 6. Use linear approximation to estimate $\sqrt[3]{700}$.

*Solution:*

© Krista King Math

Let $f(x) = \sqrt[3]{x}$ and $a = 729$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(729) + f'(729)(x - 729)$$

Find the pieces needed for the formula.

$$f\,(729) = \sqrt[3]{729} = 9.$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(729) = \frac{1}{3\sqrt[3]{729^2}} = \frac{1}{243}$$

Then the linear approximation is

$$L(x) = 9 + \frac{1}{243}(x - 729)$$

$$L(x) = \frac{1}{243}x + 6$$

Use this approximation to estimate $\sqrt[3]{700}$.

$$f(700) = \frac{1}{243}(700) + 6 = \frac{2{,}158}{243} \approx 8.88$$

© Krista King Math

## LINEARIZATION

■ 1. Find the linearization of $f(x) = x^3 - 6x^2 + 7x - 2$ at $x = 2$.

*Solution:*

The linearization formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 2$, so the linearization is $L(x) = f(2) + f'(2)(x - 2)$. Find the pieces that we need for the linearization formula.

$$f(2) = 2^3 - 6(2)^2 + 7(2) - 2 = -4$$

$$f'(x) = 3x^2 - 12x + 7$$

$$f'(2) = 3(2)^2 - 12(2) + 7 = -5$$

Plugging these pieces into the linearization gives

$$L(x) = -4 - 5(x - 2)$$

$$L(x) = -4 - 5x + 10$$

$$L(x) = -5x + 6$$

■ 2. Find the linearization of $g(x) = \sqrt{5x + 9}$ at $x = 8$.

© Krista King Math

*Solution:*

The linearization formula at $x = a$ is $L(x) = g(a) + g'(a)(x - a)$. In this problem, $a = 8$, so the linearization is $L(x) = g(8) + g'(8)(x - 8)$. Find the pieces that we need for the linearization formula.

$$g(8) = \sqrt{5(8) + 9} = \sqrt{49} = 7$$

$$g'(x) = \frac{5}{2\sqrt{5x + 9}}$$

$$g'(8) = \frac{5}{2\sqrt{5(8) + 9}} = \frac{5}{2\sqrt{49}} = \frac{5}{14}$$

Plugging these pieces into the linearization gives

$$L(x) = 7 + \frac{5}{14}(x - 8)$$

$$L(x) = 7 + \frac{5}{14}x - \frac{40}{14}$$

$$L(x) = \frac{5}{14}x + \frac{29}{7}$$

■ 3. Find the linearization of $h(x) = e^{x-2} + 4$ at $x = 3$.

*Solution:*

© Krista King Math

The linearization formula at $x = a$ is $L(x) = h(a) + h'(a)(x - a)$. In this problem, $a = 3$, so the linearization is $L(x) = h(3) + h'(3)(x - 3)$. Find the pieces that we need for the linearization formula.

$$h(3) = e^{3-2} + 4 = e + 4$$

$$h'(x) = e^{x-2}$$

$$h'(3) = e^{3-2} = e$$

Plugging these pieces into the linearization gives

$$L(x) = e + 4 + e(x - 3)$$

$$L(x) = e + 4 + ex - 3e$$

$$L(x) = ex - 2e + 4$$

© Krista King Math

## ABSOLUTE, RELATIVE AND PERCENTAGE CHANGE

■ 1. What is the absolute change of $f(x)$ from $x = \pi$ to $x = 2\pi$?

$$f(x) = 3x^2 - \cos\left(\frac{x}{2}\right)$$

*Solution:*

Absolute change is the difference between $f(2\pi)$ and $f(\pi)$.

$$f(2\pi) = 3(2\pi)^2 - \cos\left(\frac{2\pi}{2}\right) = 12\pi^2 - (-1) = 12\pi^2 + 1$$

$$f(\pi) = 3(\pi)^2 - \cos\left(\frac{\pi}{2}\right) = 3\pi^2 - (0) = 3\pi^2$$

The absolute change is the difference, or

$$f(2\pi) - f(\pi)$$

$$12\pi^2 + 1 - 3\pi^2$$

$$9\pi^2 + 1$$

■ 2. What is the relative change of $g(x)$ from $x = 2$ to $x = 3$?

$$g(x) = 2x^4 - 3x^2 - 5$$

© Krista King Math

*Solution:*

Relative change is the quotient of the difference between $g(3)$ and $g(2)$, and the value of the function at the left edge of the interval, $g(2)$.

$$g(3) = 2(3)^4 - 3(3)^2 - 5 = 2(81) - 3(9) - 5 = 130$$

$$g(2) = 2(2)^4 - 3(2)^2 - 5 = 2(16) - 3(4) - 5 = 15$$

The difference is $130 - 15 = 115$, so the relative change is

$$\frac{g(3) - g(2)}{g(2)} = \frac{115}{15} = \frac{23}{3} \approx 767\,\%$$

■ 3. What is the relative change of $h(x)$ from $x = 0$ to $x = \pi$?

$$h(x) = \tan x + 4x + 2$$

*Solution:*

Relative change is the quotient of the difference between $h(\pi)$ and $h(0)$, and the value of the function at the left edge of the interval, $h(0)$.

$$h(\pi) = \tan \pi + 4\pi + 2 = 0 + 4\pi + 2 = 4\pi + 2$$

$$h(0) = \tan 0 + 4(0) + 2 = 2$$

The difference is $4\pi + 2 - 2 = 4\pi$, so the relative change is

© Krista King Math

$$\frac{h(\pi) - h(0)}{h(0)} = \frac{4\pi}{2} = 2\pi \approx 628\,\%$$

© Krista King Math

## ESTIMATING ABSOLUTE, RELATE AND PERCENTAGE CHANGE USING DIFFERENTIALS

■ 1. Use differentials to calculate the relative rate of change of $f(x) = 4x^2 + 5x - 8$ at $x = 3$.

*Solution:*

Using differentials, the relative change of $f(x)$ at $x = a$ is calculated as

$$\frac{f'(a)}{f(a)}$$

Since $f(x) = 4x^2 + 5x - 8$ and $f'(x) = 8x + 5$,

$$\frac{f'(x)}{f(x)} = \frac{8x + 5}{4x^2 + 5x - 8}$$

$$\frac{f'(3)}{f(3)} = \frac{8(3) + 5}{4(3)^2 + 5(3) - 8} = \frac{29}{43} \approx 0.6744$$

■ 2. Use differentials to calculate the relative rate of change of $g(x) = -9 + 2x + 3x^2 - x^3$ at $x = 1.8$.

*Solution:*

© Krista King Math

Using differentials, the relative change of $g(x)$ at $x = a$ is calculated as

$$\frac{g'(a)}{g(a)}$$

Since $g(x) = -9 + 2x + 3x^2 - x^3$ and $g'(x) = 2 + 6x - 3x^2$,

$$\frac{g'(x)}{g(x)} = \frac{2 + 6x - 3x^2}{-9 + 2x + 3x^2 - x^3}$$

$$\frac{g'(1.8)}{g(1.8)} = \frac{2 + 6(1.8) - 3(1.8)^2}{-9 + 2(1.8) + 3(1.8)^2 - (1.8)^3} = -\frac{55}{27} \approx -2.037$$

## 3. Use differentials to calculate the relative rate of change of $h(x) = \arctan(2x)$ at $x = 1/2$.

*Solution:*

Using differentials, the relative change of $h(x)$ at $x = a$ is calculated as

$$\frac{h'(a)}{h(a)}$$

Since $h(x) = \arctan(2x)$ and $h'(x) = 2/(1 + 4x^2)$,

$$\frac{h'(x)}{h(x)} = \frac{\frac{2}{1 + 4x^2}}{\arctan(2x)}$$

© Krista King Math

$$\frac{h'\left(\frac{1}{2}\right)}{h\left(\frac{1}{2}\right)} = \frac{\frac{2}{1+4\left(\frac{1}{2}\right)^2}}{\arctan\left(2\left(\frac{1}{2}\right)\right)} = \frac{\frac{2}{1+4\left(\frac{1}{4}\right)}}{\arctan(1)} = \frac{\frac{2}{2}}{\frac{\pi}{4}} = \frac{4}{\pi} \approx 1.273$$

© Krista King Math

FOR THE LOVE OF MATH

WWW.KRISTAKINGMATH.COM